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Quantum Net Dynamics

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The quantum net unifies the basic principles of quantum theory and relativity in a quantum spacetime having no ultraviolet infinities, supporting the Dirac equation, and having the usual vacuum as a quantum condensation. A correspondence principle connects nets to Schwinger sources and further unifies the vertical structure of the theory, so that the functions of the many hierarchic levels of quantum field theory (predicate algebra, set theory, topology, ..., quantum dynamics) are served by one in quantum net dynamics.

1. INTRODUCTION

The Dirac equation is the most fruitful union of relativity and quantum principles we have so far, and any deeper fusion of these principles must support it. Here I discuss an alternative to the concept of the quantum field, a finite concept of quantum net (Finkelstein, 1987) that satisfies stronger principles of locality, superposition, and relativity than field theory, and supports the Dirac equation. Quantum net dynamics has a fundamental time or chronon ꓵ whose singular limit as ꓵ→0 is the transition from quantum to classical spacetime. There is a natural trial ψ vector for Vacuum I possessing a hypercubical symmetry, yet exactly Lorentz invariant, and supporting a local action principle leading to the Dirac equation.

The events of the quantum net are themselves quantum nets, and even more than the spacetime points of Snyder (1947), are treated by quantum kinematic, group, and dynamical principles, with Fermi-Dirac statistics [not Maxwell-Boltzmann as for Snyder (1947) nor Bose-Einstein as for Finkelstein (1969)].

Here are the main guiding principles of this program of theory construction, with the understanding that they evolve with the theory.

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1.1. Locality

Locality (of the Einstein kind) is the principle that fundamental concepts and laws connect events only to their infinitesimal neighborhood. It impels us to seek what these events are, and what connects them to their neighbors. Locality convinced Newton that even his own beautiful and powerful nonlocal theory of gravity was merely phenomenological rather than fundamental, leaving it to Einstein to provide the first plausible local theory of gravity. Today, with Bell's theorem, locality decides in favor of the quantum principle of superposition and against its classical reformulations, which are nonlocal theories. Thus, when it has come down to a choice of physical theories, locality has at critical junctures been given more weight even than accurate experimental predictions and common sense, and this judgment has been vindicated subsequently by further experimental successes. Locality today is combined with causality so that the neighborhoods it refers to are defined as causal neighborhoods, and applied to the action principle. The furthest form that locality has taken in this century is the gauge theory of fundamental forces. Here, far from giving up locality, I extend it. Locality has paid off so magnificiently that it is natural to explore what results if we make our other principles consist with it by "localizing" them.

1.2. Local Nonunitarity

Unitarity is a nonlocal concept, referring to an integral over all space at one time, so it cannot be fundamental. (A given unitary theory may be local; but the class of unitary theories is a nonlocal class.) It is meaningless or a contradiction in terms to ask that the fundamental local processes of nature be unitary, for unitarity applies only to the global process. Since unitarity works too well to be dropped, we localize it as follows.

1.3. Local Organization

If we accept locality, then when a phenomenological theory postulates a global symmetry principle, such as unitary or Poincaré invariance, it is up to the fundamental theory to provide a local foundation. Macroscopic structures generally exhibit less symmetry than the dynamical laws that govern them locally, as in crystallization and magnetization. The usual name "spontaneous symmetry-breaking" for this phenomenon expresses the failure of the theorist rather than the accomplishment of the experimentalist; the crystal has less symmetry than the vacuum, to be sure, but it has more symmetry and order than the melt from which it crystallizes. In this phenomenon the dynamical evolution does not break the symmetry of the system, but increases it, at the price of a decrease in order elsewhere. Moreover, in some processes of this general kind, such as biological ones, the order that increases is more complex than is expressed by a group of transformations, and symmetry is not the relevant measure of order. In general, a nonlocal order that arises spontaneously from local interactions and is maintained by them we call local organization; it is likely that this concept overlaps with the autopoeisis of Maturana, but I am not sufficiently sure to use his term. Spontaneous symmetry-breaking is a special case of local organization.

The unitarity taken as fundamental by quantum field theory, being nonlocal, cannot be fundamental in net theory; we recover it through local organization.

None of our initial principles of relativity (Poincaré invariance), superposition, and finiteness are local. Unlike unitarity, however, they may be applied at the local level, becoming local relativity, local superposition, and local finiteness. These we take as fundamental in quantum net dynamics. They are described in paragraphs that follow.

1.4. Local Finiteness

Having learned that the world need not be Euclidean in the large, the next tenable position is that it must at least be Euclidean in the small, a manifold. The principle of infinitesimal locality presupposes that the world is a manifold. But the infinities of the manifold (the number of events per unit volume, for example) give rise to the terrible infinities of classical field theory and to the weaker but still pestilential ones of quantum field theory. Further, the manifold postulate freezes local topological degrees of freedom which are numerous enough to account for all the degrees of freedom we actually observe. It is imperative to explore beyond the manifold.

The next bridgehead is a dynamical topology, in which even the local topological structure is not constant but variable and, in the most monistic extrapolation of the theory, is the only variable there is. The problem of coping with all topologies of infinitely many points is so absurdly unphysical, as well as unfeasible, that dynamical topology virtually forces us to a more atomistic conception of causality and spacetime than the continuous manifold, and to a finite locality of immediate neighborhoods rather than infinitesimal ones. In quantum net dynamics, the manifold arises as an approximate description of a spontaneously organized net which prevails in the moderate pressures and temperatures of ordinary experience, and is a necessity for life, but is contingent and not truly fundamental in nature. Local finiteness has contended against the continuum principle for millenia, but futilely since the advent of classical mechanics. I give evidence of its renewed viability in this work.

1.5. Local Superposition

The quantum principle of superposition states that the physical properties or predicates of the system form, not a Boolean algebra, but a projective geometry. Then, vectors represent maximal predicates (predicates giving maximal information). The term superposition refers to the linear combination of such vectors, which leads from determinate cases to indeterminate ones.

If fundamental principles must be local, and superposition is to be fundamental, we must expect a principle of local superposition: The independent local elements of the world and their causal connections to their neighbors are also subject to quantum superposition. Their predicates, too, form projective geometries. This seems to be a relatively new and fertile extension of the superposition principle. It is violated by quantum field theories, which superpose global ψ vectors and ψ vectors for fields at a point (if such entities make sense at all), but build spacetime from classical spacetime points, not quantum ones. In quantum field theory, for example, a point is represented by four coordinates, and any property of a point by a subset of the space of real-number quadruples $\mathbb{R}^4$. Such properties of points do not admit quantum superposition.

Local superposition means that the global quantum system must be composed of local quantum systems. This kind of composition is common in classical physics, where (say) manifolds are composed of neighborhoods, and lattices are composed of edges. Classical physics accomplishes it with the operator of set formation written $\{\alpha\}$ nowadays and $\iota\alpha$ by Peano (1888). In the classical lattice, for example, we first combine points $p, p', p'', \ldots$ into pairs $\{p, p'\}$, $\{p, p''\}, \ldots$ and then combine the pairs into the lattice $\{\{p, p'\}, \{p, p''\}, \ldots\}$, without omitting the pair structure. In Peano's more algebraic notation, this set of pairs is $\iota(\iota(p \vee p') \vee \iota(p \vee p'') \vee \ldots)$. If we combined all the points without ι's, as $p \vee p' \vee p'', \ldots$, we would lose the lattice topology. The operator ι is the vertebra of classical mathematics: In a natural formulation of the algebra of sets, ι is the sole rank-raising operation. In Peano's theory of the natural numbers, ι becomes the successor function. Here we use ι for causal succession.

The analog of the combination $p \vee p'$ in quantum theory is a tensor product, possibly symmetrized or antisymmetrized, depending on statistics. But if we combine all the quantum elements of a global system directly, say by a tensor product, we lose their topology. We must first combine them into cells, and then combine the cells without losing the distinction between them. This implies a hierarchical combination of quantum systems not used in quantum field theory or earlier forms of quantum mechanics. But the standard quantum kinematics lack any analog for ι, and are therefore forced to cling parasitically to classical host theories for their backbone, as in canonical quantization. A free-standing quantum theory needs its own ι.

Since we use elementary set theory to express the hierarchy of classical organizations, as above, I form a quantum set theory for quantum organizations with an operator analogue of the Cantor bracket $\{\cdot\}$ and the Peano ι. This quantum set theory is thus an outcome of relativistic locality and quantum superposition in their strong senses. The operator ι first appeared as a (second Fermi–Dirac) quantization operator Q in an earlier attempt at quantum set theory (Finkelstein *et al.*, 1959, Part III). In the net interpretation, ι serves as a dynamical operator as well.

A second extension of the superposition principle in this work derives, not from locality, but from intensionality. The quantum logic and set theory I use here have an important symmetry that the usual (Von Neumann) one lacks. Intensionality is the principle that every set is associated with a predicate of membership in that set; this association is an isomorphism from the algebra of sets into that of classes. The predicate is called the intension of the set, and the set of the extension of the predicate. Extensionality is the principle that every class is the intension some set, its extension. Originally assumed for classes in general, extensionality led to famous antinomies and had to be weakened; here we use it for our finite predicates without such problems.

In the usual quantum kinematics of the electron, predicates (= classes) are subspaces of the one-electron Hilbert space H_1, while a set of electrons (say the set of the electrons in a given silver atom) is maximally described by a ψ vector in the Grassmann algebra over H_1. Evidently there are many more electron sets than electron classes in quantum kinematics. This violates intensionality. Since it would be unprecedented for the quantum theory to have less symmetry than the classical, my quantum set theory uses the same Grassmann algebra for both classes and sets, restoring and enlarging the classical symmetry. Now quantum classes are subject to superposition just as quantum sets (Fermi–Dirac ensembles) are.

This fusion is crucial for quantum net dynamics and has the following curious consequence.

Second Fermi–Dirac quantization imbeds a linear space H in the Grassmann algebra over H, which I write as $\mathbf{V}H$, and maps the vectors ψ of H into generators of $\mathbf{V}H$. Since ι does these, we may call it a quantization operator. (Quantification would be a better name than quantization, since ι leads from a theory of "True or false?" to one of "How many?") Quantization is ordinarily not usable as a dynamical operator, since it enlarges the Hilbert space, but is carried out only by the theorist in setting up the kinematics, perhaps once or twice or (nowadays) thrice during the invention of a theory.

For nets, ι is a dynamical operator, something that can occur in nature, because VQET = QET. Quantization occurs at least 10^{26} times per second in the vacuum nets described below.

The preceding sentence would be a meaningless word-salad in the standard quantum theory. We can combine quantum concepts like quantization with spacetime concepts like "times per second" so freely because net theory is such an intimate union of quantum theory and relativity.

Net theory unifies two superposition processes that are usually considered to belong to different species: the addition of classical spacetime vectors, like displacements or momenta, and the addition of ψ vectors. In the nets described below these are not dinstinct processes that merely happen to be mathematically similar, but are the same process, carried out on a macroscopic ψ vector or on a microscopic one, respectively.

Local superposition provides a local linear space that permits us to formulate a quantum principle of local relativity: The Lorentz group (in its most fundamental, spinor, form SL_2) acts upon the linear space of ψ vectors describing the immediate neighbors of an event and respects their causal connection. We return to local relativity in a later paragraph.

1.6. Strong Superposition

A third extension of superposition beyond that of Heisenberg's quantum theory occurs in Schwinger (1970) source theory and is further extended here. By the strong superposition principle I mean that all physical processes have maximal descriptions that may be superposed by vector addition. When two processes cannot be so superposed, we say that a superselection rule separates them. In ordinary quantum mechanics, we superpose input processes, represented by kets, and output processes, represented by bras, but we do not superpose input processes with output ones. This is a kind of superselection rule, and it is first removed in a more global or diachronic description of physical processes, relative to which the original theories of Heisenberg and Schrödinger appear as synchronic. I explain these terms next.

1.7. Synchronic and Diachronic Descriptions

A synchronic description describes its domain instant by instant. The vectors of the Heisenberg and Schrödinger theories are synchronic descriptions, but one cannot tell from a vector of these theories what is the instant it describes. Time is specified outside their Hilbert space structure, as a parameter on which vectors or operators may depend. Synchronic theories represent an experimental input process at time 0, as well as a dynamically equivalent one at some other time, by a vector $\langle\cdot|$, in a fixed Hilbert space H, and an output process by a dual vector $|\cdot\langle$, with probabiliy amplitude represented by a symbol of the form $|\cdot\langle\cdot|$. They compress an input process that may actually be distributed over time into one initial instant. They assign input and outputs to different linear spaces, thus preventing their linear superposition and tacitly positing a superselection rule. Synchronic quantum theories describe an experiment as a dialogue between the injection group and the ejection group of the laboratory.

Moreover, these theories take it for granted that time is a manifold; relativity then makes spacetime one as well. We cannot freely explore the dynamics of the topology of time in a theory which fixes this structure by postulation, like the quantum theories. This is the original reason for using a diachronic description here; the strong superposition principle is an afterthought.

The Schwinger and Feynman quantum action principles are diachronic theories. So is Schwinger source theory (Schwinger, 1970), where an element $\langle\cdot|$ of a linear space S describes both input and output (i/o) processes over the entire experimental spacetime region, allowing their superposition and lifting the superselection law between them. The vectors of S are called sources, and describe targets or sinks as well. The distinction between input and output vectors is then made within S on the basis of the sign of the frequency; we arbitrarily assign the time dependence $e^{+i\omega t}$ to an output or target process, and $e^{-i\omega t}$ to an input or source, where $\omega > 0$. Source theory presupposes the spacetime manifold and is to emerge as a limit of net theory, which does not, as the net constant $\hbar \to 0$.

For fermions, S is an algebra with the addition operation $(+)$ for quantum superposition, a constant i that (like 1) stands for the vacuum as an element of S and for a quantum phase shift of a quarter period as a multiplier on S, and a Grassmann product $\vee$ for the combination of i/o processes. We may call the first-grade generators of S elementary sources, and the higher-grade ones composite. If the algebra S is to be free of superselection laws, then its underlying ring of coefficients must be commutative; we provisionally assume the complex numbers $\mathbb{C}$ as is usual.

There are no dynamical equations governing the sources of S. These are external sources, freely determined by the experimenter, whose dynamics is not under study.

Each vector $|\cdot\langle$ of the dual space S^{D} assigns a probability amplitude $|\cdot\langle\cdot|$ to each source $\langle\cdot|$. It therefore expresses a dynamical law or force law, and is called a field. The first-grade generators of S^{D} are dual to elementary sources and are called elementary fields. Fields, too, are subject to no dynamical equations; these are external fields. In the toy system of crossed polarizers, a source vector $\langle\cdot|$ of grade two describes both polarizers at once, and a field dual vector $|\cdot\langle$ describes (say) sugar water between them.

Diachronic theories describe the experiment as a dialogue between the external field group and the i/o group of the laboratory.

The semantics of a diachronic theory assigns vectors to experimental sources and dual vectors to experimental fields. Where a synchronic theory may suppose that every vector in its Hilbert space H represents a possible input process, a diachronic one would postulate that every vector in S and every dual vector in S^{D} represent possible sources and fields. In accord with strong superposition, we freely add vectors in S to dual ones in S^{D}, the two anticommuting with each other; that is, the most general diachronic process is maximally described by a vector in the Grassmann algebra $\mathbf{V}(S_1 \oplus S_1^D)$ over the linear space of first-grade sources and fields.

Regarded as a diachronic description, Feynman's path amplitude represents a dynamical law, not an i/o process, and therefore may be identified with an element of S^{D}, not S.

I turn now to the implementation of the above principles. The standard support for the Dirac equation in the presence of gravity today is a Bergmann (1957) or spin manifold B_2. Others include the checkerboard of Feynman and Hibbs (1965) and Feynman (1972) and the spacetime code of Finkelstein (1969). The present net theory grows from these, the causal manifold of Alexandroff (1956), and the spin net of Penrose (1971). I regard a spin manifold B_2 as a map of the world for a global observer who coordinates a network of local quantum spin experimenters. The fundamental entities of a B_2 are (see Appendix for notation):

- The metric form, or Infeld–van der Waerden form, which is a linear 2×2-Hermitian-matrix-valued form $g(v, x) = g_s(x)v^s$ with $g_s = (g_{s\Sigma^*\Sigma})$ (in many papers g is written σ).
- A spinor connection $D^{\Sigma}_{s\Sigma'}$.

These admit quantum interpretations:

- $g(v, x)$ is the quantum probability metric in the Hilbert space of spin at a spacetime point x for an experimenter with world-velocity v^s. Hence the name metric form.
- $-i\hbar D_s$ is the energy-momentum (kinetic plus potential) operator of a spin-$\frac{1}{2}$ quantum.

The proper time interval is expressed in terms of the linear metric form $g_s(v)$ by $\|dx\| = g_{ss'}\, dx^s\, dx^{s'} = \det[g(dx)]$; thus, the linear metric form $g_s(v)$ required for the Dirac equation describes gravity and the spacetime pseudometric quadratic form $g_{ss'}$ as well as the local quantum spin metric. A generalization B_N with N-component spinors (Finkelstein, 1986) is also a helpful stepping stone to net theory.

The quantum net provides a quantum theory of the B_2 spacetime, and thus of gravity, but not a canonical one. I point out an alternative to canonical quantization for gravity:

1.8. Coherent Quantization

For a quantum theory of spacetime structure, judging from experience with macroscopic matter, we must choose between two ways, incoherent and coherent, to extract macroscopic classical variables from microscopic quantum ones as $\hbar \to 0$. In both, large quantum ensembles produce macroscopic classical behavior, but these ensembles may be incoherent (like the Thomas-Fermi model of the atom) or coherent (like superfluids). In the incoherent case, a probability distribution ρ emerges as the classical variable; in the coherent, a probability amplitude distribution ψ. We may call the respective inverse processes, going from macroscopic theory to quantum, incoherent and coherent quantization, according as they start without or with quantum phase data.

I am driven to a coherent quantum scheme by a persistent problem with any incoherent one: In any construction of spacetime from spinors, spacetime vectors do not transform as spinor statistical operators ρ^A_B under SL_2, but as pair amplitudes ψ_{A*B}; not as $D(1,0) \oplus D(0,0)$, but as $D(\frac{1}{2}, \frac{1}{2})$. I understand this now as follows.

Canonical quantization undoes the classical limit. Canonical quantization of spacetime structure (or of a harmonic oscillator or a hydrogen atom) would yield a system having the usual spacetime (or oscillator or atom) as a high-quantum-number excitation. In the classical limit of many excitations we lose all quantum phase information. Canonical quantization thus begins from a theory without quantum phases; it is an incoherent quantization. The underlying quantum system "heats up" to the classical one.

In superfluidity and superconductivity, we meet macroscopic systems with low quantum numbers, not high ones. Their macroscopic variables—the current in the superconductor, the velocity in the superfluid, the potential difference of the Josephson junction—carry quantum phase information, preserved because the underlying system "freezes" to the macroscopic one. To recover the quantum system, we must use the phase information carried by the macroscopic description; this is coherent quantization.

Nambu long ago pointed out that the physical vacuum is likely a quantum condensation, a low-quantum-number limit, and this is now rather widely accepted; for example, the latent heat of this phase transition is supposed to drive the inflation of the early universe. In quantum net dynamics, spacetime structure, too, is a macroscopic quantum effect; the quantum net freezes to our usual spacetime rather than heating up to it. We do not quantize spacetime canonically, any more than one does the hydrodynamics of liquid helium II. We construct a quantum dynamical system whose macroscopic quantum ψ vectors define a classical spacetime. We start from ψ's carrying quantum phase information; this is a coherent quantization. There are systematic procedures for canonical quantization, but not for coherent.

Here I quantize spacetime structure by giving a quantum spacetime whose coherent macroscopic kets define a B_2 in the classical continuum limit. Henceforth in this paper, quantization means coherent quantization.

The simplest finite (and hence necessarily nonunitary) way to express local relativistic invariance, local causality, and local quantum superposition is not by a field theory, but by a net of dynamical relations with a fundamental time Π.

To explore this net as an alternative to the continuum, I give a trial vacuum net approaching Minkowski spacetime in the continuum limit and, like B_2, designed to support Dirac's equation. Its regular crystallike structure recalls Feynman's checkers game for the two-dimensional Dirac equation, as well as the crystalline aëther of Newton's *Opticks.* It has coordinate and derivative operators x^s and ∂_s defined for all values of the fundamental time Π, but normal ("observables") only in the continuum limit. The global unitary structure of the continuum quantum theory emerges from this nonunitary net theory in a nonuniform approach to the classical continuum.

The operators of quantum set theory have the syntax of a relativistic quantum theory, but we do not have a quantum theory until we connect these operators to physical determinations. It is useful to do this by a correspondence principle connecting the net theory in the continuum limit to the Feynman–Schwinger action principle.

An action principle for nets results from quantizing the spacetime in the continuum action principle. As an exercise in net dynamics, and to test its viability, from the action principle for the Dirac equation I derive a covariant local action for holes in the vacuum net leading to a Weyl or Dirac equation in the classical continuum limit.

2. THE CONCEPT OF THE NET

First I present a classical theory of nets, and then I quantize it (coherently, as has already been stipulated).

2.1. Classical Net

In the most familiar formulation of a causal structure, the elementary entities are point events $\alpha, \beta, \ldots$ and they support a causal relation $\alpha C \beta$, a set of pairs. At once locality compels us to reject C as fundamental variable in favor of its germ, the local connection relation $\alpha c \beta$, "α connects to β" (it being understood that this connection is meant to be immediate and directed from cause α to effect β).

I am guided at this point (and several others) by the superposition principle: Fundamental descriptions are coherent. (In Von Neumann's parlance, they are pure cases.) This means they give maximal information. In this sense statistical mechanics is less fundamental than quantum mechanics, the Birkhoff–Von Neumann quantum logic of $\cap$ and $\cup$ is less fundamental than the usual Hilbert space theory, and dissipative processes are less fundamental than isentropic ones. (This does not mean less true. Incoherent descriptions are safer bets than coherent ones just because they give less information.)

In the present application, since equations give more information than inequalities, coherence impels us to seek an equational theory, not a relational one. We take as fundamental not the causal *relation* $\alpha c \beta$, which says little about β, as in Finkelstein (1969), but the dynamical *equation*

$$\beta = \iota(\delta, \alpha) \tag{1}$$

stating that event β is the successor of dynamical process δ and event α, which defines β completely in the classical theory and maximally in the quantum one. This change in expression from relations to equations makes no difference in the classical theory, but it affects the quantization: we quantize ι, not c. I represent equation (1) by a figure such as Figure 1, which represents an event and two possible successors.

Let SET be the algebra whose elements are sets and whose operations are a monadic operation ι and a dyadic one $\vee$, graded by cardinality: $\text{SET} = \text{SET}_0 \cup \text{SET}_1 \cup \text{SET}_2 \cup \ldots$, where SET_0 consists solely of the null set, SET_1 consist of all monads (unit sets), SET_2 of all pair sets, ... Here "set" means "hereditarily finite set", a finite expression in the three symbols

- 1 null set
- ι set operator (Peano, 1888); $\iota\alpha$, also written $\{\alpha\}$, is the monad (unit set) of α; ι: $\text{SET} \leftrightarrow \text{SET}_1$ is bijective
- $\vee$ disjoint union (Peirce, 1868*a,b*)[2]: $\alpha \vee \beta = \alpha \cup \beta$ when $\alpha \cap \beta = 1$; otherwise 0 (undefined); commutative, associative, and with identity 1

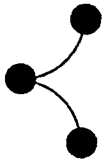

Fig. 1. A binary node.

[2]My 0 and $\vee$ are Peirce's ∞ and $+$. My nets recall his logic diagrams.

and parentheses or Polish notation. The number of nested $\{\cdot\}$'s in a set is called its rank. The number of monadic factors is its grade.

Interpretation. We express any net as a set by associating monads $\alpha, \beta, \ldots$ with events and writing

$$\beta = \{\alpha_1 \vee \ldots \vee \alpha_N\} = : \iota(\alpha_1 \vee \ldots \vee \alpha_N) \tag{1'}$$

to mean that *the event β is the successor of the α.*

We call equation (1) an N-ary node if there are N independent possible δ's. Every set defines a net of such nodes. There are no causal loops in any net, nor two distinct events with the same inputs. The group of the N-ary node (1) is S_N, the symmetric group on the δ's.

2.2. Quantum Net

The quantization of SET is a generative Grassmann algebra QET $=$ $\text{QET}_0 \oplus \text{QET}_1 \oplus \ldots$ graded by cardinality, whose elements are the kets of the quantum set, or qets for short. A qet is a finite expression in

- i imaginary unit.
- $\vee$ Grassmann product; commonly $\wedge$, but see Peano (1888)
- ι qet operator, also written $\langle\cdot|$; ι: QET $\rightarrow$ QET_1 is antilinear and bijective
- $+$ quantum superposition, Grassmann addition

The top three symbols in this list correspond in interpretation to those of SET; $\iota = \langle\cdot|$ makes QET self-generating; $+$ with its quantum interpretation makes QET a quantum theory.

The qet bracket $\langle\cdot|$ unifies Cantor's set bracket $\{\cdot\}$, Dirac's ket bracket $\langle\cdot|$, and Grassmann's extensions, in that qets add like kets, nest like sets, and multiply like extensions, as in equations (2_1), (2_2), and (3) below. Dual qets are written $|\cdot\langle$, so that operators $\langle\cdot|\cdot\langle$ look like arrows, and transition amplitudes $|\cdot\langle\cdot|$ like numbers.

Interpretation. I propose that QET describes a universal quantum entity, in the sense that SET does a universal finite mathematical one. Here I give QET a specific interpretation: A qet describes a quantum net of dynamical nodes completely in Bohr's sense of the word, and incompletely but maximally in Von Neumann's. It is a plausible thesis that set theory is a universal language for mathematics, and finite set theory for finite mathematics. Therefore it may sound disappointingly trivial to propose that the physical world may be well described by quantum sets. How could it not be? But usually the same set symbols are given an enormous number of different interpretations in physics, according to need; just consider, say, the variety of physical quantities that are represented by numbers. The actual language is then not set theory, but a much richer one, with many logically independent concepts, each with its own label from outside set theory. Here we give each of the three set symbols 1, $\vee$, ι a single physical interpretation in all its occurrences. This is a genuine unification.

To make it work I have had to adapt the set theory slightly. Our physical set theory does not have exactly the same primitives $\varnothing$, $\cup$, $\in$ as the usual mathematical set theory. The null set $\varnothing$ survives as 1, but $\cup$ and $\in$ have been replaced by $\vee$ and ι to meet the needs of quantum physics. I have modified quantum logic as well as classical set theory to produce this fusion, by restoring intensionality.

The Grassmann operations on qets have the usual quantum meanings given to them in the theory of many fermions, which are natural extensions of the classical meanings of the corresponding operations on sets. Read now as an equation among qets, (1) means that the successor of the quantum events δ and α is β. Qets form a Grassmann algebra graded by cardinality and generated by the operator ι.

The group of the N-ary node (1) is now SL_N, and that of a binary node where $N=2$, is SL_2. This SL_2 is promoted to the global SL_2 of special relativity by the quantum condensation of Section 3, and presumably serves as the local structure group of the B_2 arising in the classical continuum limit from defective nets. Qet brackets quantize in the second Fermi-Dirac, mapping any qet, whether even or odd in grade, into an independent first-grade qet.

The quantum kinematics of nets is expressed by $+$ and $\vee$, and local causality by $\vee$ and ι. In the usual quantum theory these principles are assigned to different levels of theory. The synthesis of relativity and quantum kinematics in QET permits the free intermixing of these principles.

3. THE VACUUM

What appears as the law of nature in synchronic theories is the vacuum field in diachronic ones. We give a vacuum net now, as a space κ of kinematically allowed paths, and a subspace Δ of dynamically allowed paths.

3.1. Vacuum I

We arbitrarily model the vacuum with a qet, not a dual qet. To build κ and Δ, we form $\kappa(Z')$ and $\Delta(Z')$ corresponding to a terminated future cone of some fiduciary event $\psi \in$ QET far in the past. $\kappa(Z')$ comprises qets no more than Z' above ψ in rank. Then we let $Z' \to \infty$. To explore the continuum limit, we simultaneously let $Z' \to \infty$, $\hbar \to 0$ in such a way that

Z'ת remains constant at the macroscopic time coordinate t of the reference event ψ.

First I infer from relativistic SL_2 that events have two possible successors. Let those to ψ be designated by ψ^{Σ}, and write the two δ's as $\langle\uparrow|$, $\langle\downarrow|$.[1] Then by (1)

$$\psi^{\Sigma} = \langle \delta^{\Sigma} \vee \psi | \tag{2_1}$$

Second, from the role of the vector representation $D(1/2, 1/2)$ of SL_2, I infer that the two successors to the ψ^{Σ} transform as $\psi^{\Sigma\Sigma^*} = \psi^{\sigma} = \langle \Sigma\Sigma^* |$, so that, by (1), using the antilinearity of ι

$$\psi^{\Sigma'\Sigma^*} = \langle \delta^{\Sigma'} \vee \psi^{\Sigma} | \tag{2_2}$$

The ψ^{σ} and ψ are the five events of the elementary quantum future null cone (Figure 2). [This replaces the excessively symmetric pentacle of Finkelstein and Rodriguez (1984).]

I iterate this process Z' times, alternating spinor inputs proper and conjugate spinor inputs, and thus forming a sequence (2_1), (2_2), $\ldots$, $(2_{Z'})$, defining a tree of events with integer-spin qets $\psi^{(\sigma)} = \langle (\sigma) |$ connected through intermediate integer-and-a-half-spin qets of the form $\psi^{(\sigma)\Sigma}$. [Compare the lattice fermion of Susskind (1977).] These make up κ.

Finally, from the macroscopic observable nature of spacetime vectors, I infer that a large number of such quantum modules condense into a four-dimensional quantum lattice. To construct an example of such D, I Hermitianize and symmetrize the $\langle (\sigma) |$ in the collective index (σ). There are 4^Z symbols $\langle (\sigma) |$ with Z 4-valued σ's, but there are only a number of independent symmetric qets $\langle \{\sigma\} |$ given by the binomial coefficient ${}^{Z}C_4 = O(Z^4)$, naturally represented by points of a sector of a 4-dimensional cubical lattice coordinatized by the occupation numbers for four values of σ. Thus, symmetry cuts the growth of the tree of qets from exponential 4^Z to 4-dimensional.

Each module inputs to four nearest-neighbor modules with nondecreasing occupation numbers. These four linearly independent inputs will give rise below to the four dimensions of spacetime.

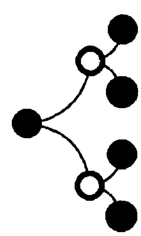

Fig. 2. Module of the vacuum net. Dots stand for events and represent independent generators of the Grassmann algebra QET. Each of the three nodes in this net represents an equation of the form $\beta = \iota(\delta \vee \alpha)$ giving its output β in terms of its inputs α and δ'. Representations $D(1/2, 0)$ (open dots) and $D(0, 1/2)$ (solid dots) alternate from right to left.

The qet $\psi^{\{\sigma\}}$ has $2Z$ spinor indices. Taken for all Z up to some maximum value Z', the $\psi^{\{\sigma\}}$ and $\psi^{\{\sigma\}\Sigma}$ span a subspace $\Delta(Z')$ of QET invariant under SL_2, corresponding to a bounded part of the past cone through ψ. $\Delta = \Delta(\infty)$ is the analogue of the future timelike paths through ψ.

The qet $\langle \mathrm{vac}\, Z'|$ is taken to be the product of all the ψ's of $\Delta(Z')$ with $Z \leq Z'$ in any convenient order. This is a Grassmann element representing $\Delta(Z')$.

Δ is isomorphic to $\mathbb{R}^{\{\sigma\}}$, and $\Delta(Z')$ to the segment of $\mathbb{R}^{\{\sigma\}}$ consisting of tensors of degree not greater than Z', and so both are SL_2 invariant.

Let Δ_σ and Γ^σ be the standard Bose-Einstein destruction and creation operators on $\{\mathbb{R}^\sigma\}$, transformed by this isomorphism to act on Δ, with $[\Delta_\sigma, \Gamma^{\sigma'}] = \delta_\sigma^{\sigma'}$. More explicitly:

Let the null qet 1 represent the initial event in the remote past, and let $\lambda^\Sigma = \langle\Sigma|$ to be fixed "link" qets with independent conjugates $\delta^{\Sigma^*} = \langle\Sigma^*|$. Define the symmetrized event-pair creation operator[1] Γ^σ by

$$\Gamma^\sigma \xi := \Gamma^{\Sigma\Sigma^*}\xi := S_+ S_H \langle \delta^{\Sigma^*} \vee \langle \delta^\Sigma \vee \xi \| \tag{3}$$

The Γ^σ generate an algebra $A(\Gamma)$ isomorphic to $\mathbb{R}^{\{\sigma\}}$. Let Δ_σ be the usual destruction operators upon $A(\Gamma)$ obeying $[\Delta_\sigma, \Gamma^{\sigma'}] = \delta_\sigma^{\sigma'}$. Then for any $\{\sigma\}$ we set

$$\psi^{\{\sigma\}} := \prod_{\sigma \in \{\sigma\}} \Gamma^\sigma \langle \varnothing | \tag{4}$$

with $2Z'$ events in succession. This completes the construction of Δ.

This construction does not explain the four-dimensionality that space-time seems to have, for its generalization from $N=2$ to arbitrary N is regrettably simple. Presumably the dynamics selects $N=2$ and makes nets with $N = 1, 3, 4, \ldots$ unstable today.

Such a quantum condensation accounts for the following features of standard physics, which would otherwise be incomprehensible in net theory.

Supermobility. From the beginning the laws of inertia and momentum conservation have been problems in net theory, since the net is not invariant under translation. Now we regard these as typical macroscopic quantum effects. Since Newton's first law implies that the mobility, usually defined as $(\partial[\text{force}]/\partial[\text{velocity}])^{-1}$ at zero velocity, is infinite in zero-temperature vacuum, we may call this macroscopic quantum effect *supermobility*. In the present theory supermobility belongs to the same family as the other two macroscopic quantum phenomena, superfluidity and superconductivity. Since the event pairing occurs between neighbors in time space rather than momentum space, the vacuum in thermal equilibrium is presumably not a two-fluid system like liquid He II.

Macroscopic Vectors. Momenta and gauge vector fields are macroscopic dynamical variables and yet have to be made up out of microscopic quantum Ψ vectors like the spinors δ^{Σ} of each node, which are not. Such "condensation" of ψ vectors into macroscopic variables occurs in the other two main superflows. Here spinors δ^{Σ} and antispinsors δ^{Σ^*} must pair and condense to vectors $\delta^{\Sigma\Sigma^*}$. All physical time-space vectors v^s are regarded as macroscopic ψ vectors $v^{\Sigma\Sigma^*}$ of condensed aggregates of Σ-Σ^* pairs, present only in the low-temperature phase, Vacuum I. The usual Minkowski coordinate and derivative operators may be modeled in net theory by the creation and annihilation operators for such pairs, the limit $\hbar \to 0$.

Metric Form and Unitarity. The linear metric form $g_{s\Sigma^*\Sigma}$ describing gravity associates a macroscopic Ψ vector $dx_{\Sigma\Sigma^*} = g_{s\Sigma^*\Sigma} dx^s$ with each physical time-space vector dx^s. This serves as the quantum metric for local spin experiments of an experimenter with world velocity $dx^s/d\tau$. The reduction of the special linear group to the unitary group may also be understood as a spontaneous symmetry breaking.

This four-dimensional hypercubical net may be regarded as a synthesis and extension of Feynman's two-dimensional checkerboard (Feynman and Hibbs, 1965) and Penrose's (1971) two-dimensional spin net.

3.2. Continuum Limit

Under Lorentz transformations the events of the quantum net do not simply permute like classical events but undergo quantum superposition. A monadic qet is about as much like a cell in spacetime × momentum-energy as a harmonic oscillator ket is like a cell in phase space. It is exactly local, Lorentz invariant, and finite in a way unlike any set in classical spacetime. For $\hbar > 0$ we can model neither K nor Δ by paths in Minkowski spacetime.

Yet the Minkowski spacetime of the continuum quantum theory with all its problems must reemerge as $\hbar \to 0$ and $Z' \to \infty$ with constant $Z'\hbar$ fixed as the macroscopic time coordinate of the reference event ψ. The paradox of the Hilbert space metric is the key to the recognition of ordinary Minkowski spacetime and its coordinates and derivations $x^{\Sigma\Sigma^*}$, $\partial_{\Sigma^*\Sigma}$ within QET. For any finite $\hbar$, the space $\Delta(Z')$ supports a finite-dimensional representation of SL_2 and thus admits no SL_2-invariant Hilbert metric. Yet as $\hbar \to 0$, $D(Z')$ must approach the Minkowski future cone $C(\psi)$ with the SL_2 invariant L^2 metric on complex functions $\Psi(x^s)|x^s \in C(\psi)$.

The resolution is that a nonunitary structure can approach a unitary one nonuniformly in the continuum limit. Replace each basis qet $\psi^{\{\sigma\}}$ in the expansion of any qet $\langle\Phi| \in D(\infty) \leftrightarrow \mathbb{C}^{\{\sigma\}}$ by the corresponding symmetrized tensor power $x^{\{\sigma\}}/\text{ח}^Z$ of a 2×2 matrix variable $x/\text{ח} = (x^\sigma/\text{ח}) = (x^{\Sigma\Sigma^*}/\text{ח})$; the time ח gives x the dimensions of a time when ψ is dimensionless. Then $\langle\Phi|$ becomes a polynomial $\Phi(x)$ in the Hermitian matrix x of degree $\leq Z'$. The linear isomorphism $\langle\Phi| \leftrightarrow \Phi(x)$ maps $\Gamma^\sigma \leftrightarrow x^\sigma/\text{ח}$ (the left multiplication operator) and $\Delta_\sigma \leftrightarrow \text{ח}\partial_\sigma = \text{ח}\partial/\partial x^\sigma$. As $\text{ח} \to 0$, $Z' \to \infty$ and $\Phi(x)$ may nonuniformly approach a holomorphic function of x in $L^2(C(\psi))$. For such functions we may define a Lorentz-invariant Hilbert norm $\|\Phi\| = \int (dx)|\Phi(x)|^2$. For any finite Z', Φ is a polynomial and $\|\Phi\|$ diverges.

As $\text{ח} \to 0$, x^σ and $-i\partial_\sigma$ become Hermitian coordinate and energy-momentum vectors. No invariant norm exists for $\text{ח} > 0$. The operators x^σ and $-i\partial_\sigma$ exist for all ח and for both $\psi^{\{\sigma\}}$ and $\psi^{\Sigma\{\sigma\}}$, obey the Heisenberg commutation relations exactly for all ח, but are not Hermitian for $\text{ח} > 0$.

Relative to dimensionless coordinate volume $(dx) = dx^{11^*} \vee \cdots \vee dx^{22^*}$ the event density ρ_{I} in Vacuum I for $T \ll T_C$ is proportional to the usual density:

$$\rho_I \to \text{ח}^{-4}(-\det g_{ss'})^{1/2} \qquad T/T_C \to 0 \tag{5}$$

where $g_{ss'}$ is the pseudometric quadratic form. The fundamental time ח appears because $g_{ss'}$ has units $(\text{time})^2$, while ρ_{I} is a pure number.

The key element of manifold spin structure for us is the local structure group SL_2 of a B_2. We identify this with the group SL_2 of a binary node, and the metric form $g_{s\Sigma\Sigma^*}$ in $\langle\text{vac}|$ with the Pauli matrices.

4. DYNAMICS

The algebra Δ is our quantum version of a causal spacetime. We turn now to the dynamics of particles on this background. The infinite hypercubical lattice of basis qets $\psi^{\{\sigma\}}$ is our correspondent to the Feynman checkerboard (Feynman and Hibbs, 1965). What moves on our board, however, are not separate men, but defects in the board itself, and they therefore do not move in an independent manner, but a way already determined by the board itself. Each phenomenological dynamics, such as those of gravity and the electroweak and strong forces, is now a further clue to the actual structure of the vacuum net. Here we see what the Dirac equation might tell us about the net. In this second stage of exploration we still use only operators in the subalgebra D of QET, but we revise $\langle\text{vac}|$.

We begin by seeking a correspondence principle connecting nets with the usual continuum dynamics.

Since our basic variables are all Grassmann qets, we seek a correspondence with the Schwinger dynamical principle for a complex Grassmann field $\psi(x)$, $\psi^*(x)$ in classical spacetime. The presentation of DeWitt (1984) for the more general superfields may be specialized to our Grassmann fields by discarding the zero-grade parts (his bodies) of his fundamental field (history); our fields are, in his terms, pure soul. Since the net is a global entity, I review the algebraic structure of global relativistic ("third-quantized," "functional") diachronic theories.

4.1. Diachronic Quantum Theories

A source in S maximally describes the i/o process relative to the system under study (Section 1.7). The principle of superposition requires S to be a module over some ring of coefficients. S is to be a Grassmann algebra, so (to avoid superselection rules) the ring must be commutative; the complex numbers $\mathbb{C}$ are convenient for the time being. Anticipating a correspondence to be established below, I write a source in S as a dual vector $|\cdot\langle$. First-grade generators of S are called elementary sources.

A vector $\langle\cdot|$ dual to a source assigns a transition amplitude to each i/o process; it therefore serves as a force law or law of motion, and is called a field. (These are external fields.) In elementary quantum physics fields modify the potential energy but not the kinetic energy. Then we learn that the most intimate parameters of the system, such as its charge and mass, may be modified by suitable fields. In diachronic theories the entire dynamical law is regarded as a field. I write F for the space of fields. The first-grade generators of F are called elementary fields. Each of F and S is included in the dual of the other.

QET is a Grassmann algebra with causal structure. So are the S and F of continuum fermionic source theory; their causal structure is that of the underlying Minkowski spacetime. Therefore I suppose that QET corresponds to S or F. We decide which by our locality postulates: The i/o processes may be as nonlocal as we please, but the vacuum field is the exponential of the action integral, which is generally postulated to be a local function of the elementary fields. Since I have given a local construction for a vacuum qet, I suppose now that qets correspond to fields, not sources; and dual qets to sources, therefore.

In diachronic Grassmann theories a source $|\cdot\langle$ is a Grassmann polynomial in elementary sources η_y with index y, which may be regarded as components of an elementary source vector η. Each η_y is an independent i/o process of the experimenter, their Grassmann product $\vee$ is their joint operation or disjoint union, and their sum $+$ is quantum superposition. For a complex theory we adjoin independent conjugate sources η^*. Dually, a field $\langle\cdot|$ may be written as a polynomial in dual generators ψ^y called elementary fields; the symbols and concepts of η and ψ are those of Schwinger (1970). A common locality postulate is that the field $\langle\text{vac}|$ may be factored as

$$\langle\text{vac}| := \rho e^{iS}$$

with $S = S(\psi, \psi^*)$ and $\rho = \rho(\psi, \psi^*)$ local in the fields and formally real.

When we further specialize this general diachronic Grassmann scheme to the standard field-theoretic one in a continuous spacetime, the index y consists of a spacetime point x and internal variables, and fields ψ^y cover the experimental spacetime region as the index y varies. The Grassmann algebra is then a continuous one, with independent generators for every point of Minkowski spacetime, an ill-defined concept optimistically indicating a limiting process to be constructed. In the continuum theory the Grassmann identity element 1 in S (which has first-grade component 0) then represents shielding the experimental region from all i/o processes.

The fields ψ^y are even more local than the usual local fields of synchronic theories. Dynamical equations relate usual fields at one point to those at another; ψ's at different spacetime points are independent variables subject to no dynamical equations because they represent actions of the experimenter, whose dynamics is left out of the system under study. To express this difference one calls the ψ's ultralocal and external. Causal relations among the ψ's are inherited from the underlying Minkowski spacetime.

We now perform a Berezin Fourier transform $\mathscr{F}:\langle\text{vac}| \to |\mathscr{F}\text{vac}\rangle$ of the field $\langle\text{vac}|$ integrating with a volume element $\prod_y d\psi^y = (d\psi)$. The $\mathscr{F}$ changes variables from the ψ's to dual Fourier transform variables, which are therefore η's, which anticommute with the ψ's. Then

$$|\mathscr{F}\text{vac}\rangle := \int (d\psi \vee d\psi^*)\rho \exp(iS + i\eta^*\psi + i\psi^*\eta) \tag{6}$$

This is Feynman path-integral form of the Feynman–Schwinger action principle, and appears here as a definition of $|\mathscr{F}\text{vac}\rangle$. The term $S_\eta := \eta^*\psi + \psi^*\eta$ in the exponent of (6) is the source term, $S + S_\eta$ is the classical action, and $\rho = \rho(\psi, \psi^*)$ is the measure density. In the continuum theory, (6) is not a well-defined expression, but the starting point of a renormalization program aimed at defining it.

Complex-valued, antisymmetric, Z-particle propagators

$$G[y_1, \ldots, y_Z] = G([y])$$

in the vacuum are given by

$$\begin{gathered} G([y]) := |\mathscr{F}\text{vac}\rangle\langle(\partial/\partial\eta)^{[y]}\langle\eta = 0| = |\eta = 0\rangle\langle\psi^{[y]}\langle\text{vac}| \\ |\eta = 0\rangle := \mathscr{F}\langle\eta = 0| \end{gathered} \tag{7}$$

where the ket $\langle\eta = 0|$ is such that for any function $f(\eta)$, $|f(\eta)\rangle\langle\eta = 0| = f(0)$; and $|\eta = 0\rangle$ is its Fourier transform.

The special system of left-handed fermions in a B_2 described by a Weyl spinor field $\psi^{\Sigma}(x)$ has the source term and derivative (kinetic) term

$$
\begin{aligned}
S_\eta &= \int (\delta x)(\psi^{\Sigma^*}\eta_\Sigma + \eta_{\Sigma^*}\psi^\Sigma) \\
S_\partial &= \int (dx)\psi^{\Sigma^*}\partial_{\Sigma^*\Sigma}\psi^\Sigma
\end{aligned}
\tag{8}
$$

where $\partial_{\Sigma^*\Sigma} = g^s_{\Sigma^*\Sigma}\partial_s$; for the generalization of S_∂ from B_2 to B_N see Holm (1989). These terms are responsible for the leading terms in the Weyl and Dirac equations, and, through the metric form g, give primary information about spacetime structure. I confine attention to them for now and leave particle interactions and masses for later.

We now require the corresponding concepts in quantum spacetime. I use "corresponding" in the sense of Bohr, but now for the limiting process $\sqcap \to 0$ rather than $\hbar \to 0$. I construct $\langle \text{vac}|$ to allow for the propagation of defects according to the Dirac equation.

For the reason given in Section 1, let us suppose that the entities corresponding to fields and sources for nets are qets and dual qets. For the derivatives and integrals with respect to fields we now require the following algebra.

4.2. Berezin Calculus of Nets

As usual, we imbed the Grassmann algebra QET in the (Clifford) algebra A(QET) as left multiplication (creation) operators, and the dual space QET^{D} in A(QET) as derivation (destruction) operators. As ξ ranges over a basis of QET_1, $\partial\xi$ ranges over the dual basis of QET_1^{D}. Then $\iota\xi = \langle\xi|$ is a Grassmann variable independent of ξ, like ξ^*.

Therefore

$$\int d\xi\,\xi = \int d(\iota\xi)\,\iota\xi = 1, \quad \int d(\iota\xi) = \int d\xi = \int d\xi\,\xi^* = \int d\xi\,\iota\xi = \int d(\iota\xi)\,\xi = 0$$

4.3. Quantizing the Quantum Action Principle

Suppose that fermions are defects in the net. We quantize the classical spacetime continuum action principle so that the continuum fermion annihilator $\psi(x)$ corresponds to a defect annihilator, that is, an event creator, $\psi^{\{\sigma\}\Sigma} = \langle\{\sigma\}\Sigma|$. This leads to the following natural replacements in the general principles (6) and (7) and in the specific action (8):

In (6) and (7)	In (8)
fields $\psi, \psi^* \to$ qets ψ, ψ^*	$\partial_{\Sigma^*\Sigma} \to \Delta_{\Sigma^*\Sigma}$
sources $\eta, \eta^* \to$ dual qets η, η^*	$\psi^{\Sigma}(x^{\sigma}), \psi^{\Sigma^*}(x^{\sigma}) \to \psi^{\{\sigma\}\Sigma}, \psi^{\{\sigma^*\}\Sigma^*}$
ket $\langle \text{vac}\vert \to$ qet $\langle \text{vac}\vert$	$S_\eta \to S_\eta = \eta_{\{\sigma\}\Sigma}\psi^{\Sigma\{\sigma\}} +$ conjugate
x integration $\to \{\sigma\}$ contraction	$S_\partial \to S_\Delta = \psi^{\Sigma^*}_{\{\sigma^*\}}\Delta^{\Sigma\{\sigma\}}_{\Sigma^*\Sigma\psi} +$ conjugate
$\partial_\eta \to \partial_\eta$	

We lower indices $\{\sigma^*\}$ in S_∂ with the Levi-Civita tensor $\varepsilon_{\Sigma\Sigma'}$, committing us to $N=2$ for now. The contraction of σ with σ^* uses the Hermitian symmetry of σ. Then (6) becomes

$$|\mathscr{F}\text{vac}\langle = \int (d\psi \vee d\psi^*)\rho \exp(iS_\Delta + iS_\eta) \tag{9}$$

and the Dirac vacuum field is not the previous $\langle\text{vac}|$, but

$$\langle \text{vac}| = \exp i[\psi^{\Sigma^*}_{\{\sigma^*\}}\Delta_{\Sigma^*\Sigma}\psi^{\Sigma\{\sigma\}} + \eta_{\{\sigma\}\Sigma}\psi^{\Sigma\{\sigma\}} + \text{conj} + S'] \tag{10}$$

where S' includes omitted mass terms and couplings.

Unlike (6), the new action principle (8) is a well-defined finite algebraic expression for any finite spacetime volume Ω, in a Grassmann algebra over a linear space of finite dimension $\sim(\Omega/\Pi^4)$, with no limiting processes to be performed. The Grassmann unit 1, the first term in the power series for the exponential, now represents, not a totally shielded experimental region, but the null set. The maximal experiment or source creates all spacetime nodes of the experimental plexus and annihilates them back down to the null set. The previous vacuum net is represented by the last terms in (10), those which create the maximum possible number of nodes in D subject to the exclusion principle.

4.4. Deduction of Dirac Equation

Obviously the spinor substructure of $\langle\text{vac}|$ can support a correspondent of the Dirac equation. A sketch suffices for now. Dirac four-component spinors enter the vacuum net in much the usual way: The destruction operator $\Delta_{\Sigma^*\Sigma}$ maps the linear space $\mathbb{C}^{\{\sigma\}\Sigma}$ into a different space $\mathbb{C}^{\{\sigma\}}_{\Sigma^*}$, not into itself. An operator that works within one linear space is particularly useful for forming propagators. This linear space is the usual direct sum $\mathbb{C}^{\Sigma}\oplus\mathbb{C}_{\Sigma^*} = \mathbb{C}^{\alpha}$ with Dirac index α.

We imbed $\mathbb{C}^{\{\sigma\}}$ naturally within $\mathbb{C}^{\{\alpha\}}$, and raise and lower α's with * and the Pauli metric

$$\beta = (\beta_{\alpha^*\alpha'}): \quad (\xi^{\Sigma}\oplus\eta_{\Sigma^*}) \to (\eta_{\Sigma^*}\oplus -\xi^{\Sigma})$$

which uses implicit 2×2 Levi-Civita ε's. Then the destruction operator $\Delta_{\Sigma^*\Sigma}$ induces an operator $\Delta = (\Delta^{\alpha}_{\alpha'})$ on $\mathbb{C}^{\alpha\{\alpha\}}$. Here, Δ is a reticular correspondent to the chiral Dirac differential operator $\partial_+ = \frac{1}{2}(1-\gamma_5)\gamma^2\partial_s$.

We may then use the Dirac operator formalism to complete the square in $S_\Delta + S_\eta$ in the standard way. Let $S = (S^{\alpha\{\alpha\}}_{\alpha'\{\alpha'\}})$ be a spinor kernel inverse to $\Delta^{\alpha}_{\alpha'}$. The propagator $W(\eta)$ for S_{net} is then given by

$$\exp[iW(\eta)] = \exp[i\eta_{\alpha\{\alpha\}} S^{\alpha\{\alpha\}}_{\alpha'\{\alpha'\}} \eta^{\alpha'\{\alpha'\}}] \det(S^{-1})$$

Away from the source η the propagator W obeys $\Delta W = 0$, which becomes the massless chiral Dirac equation in the continuum limit $\sqcap \to 0$. Events of integer spin $\psi^{\{\sigma\}}$ similarly support the complex-conjugate chiral Dirac operator, which provides a spinor of opposite chirality; the two together make a whole Dirac spinor. The plexic correspondent of the Dirac mass term $S_M = M \int (dx) \psi^* \beta \psi$ is then easy to supply.

4.5. Internal Particle Symmetries

To convey the philosophy presently underlying this approach to nature, let me sketch a tentative program for understanding the internal particle symmetries in net dynamics. Fortunately these all seem to be gauge symmetries, and may therefore act on defects in the vacuum net, in the way that the Burgers vector does on defects in a crystal. There are two approaches, phenomenological and fundamental.

Phenomenological Approach. Since we already have the net correspondent to spin, and none to charge or the other coupling constants, the first step will likely be a theory of the transport of spin in the vacuum net. That should constitute a quantum theory of the gravitational field, which is the curvature of the spin connection.

The next work is to extend the gauge theory of nets from gravity to the other interactions. This will require us to assign defect transformations to the known internal symmetry generators. Each of the known phenomenological interactions can be transplanted from the spacetime continuum to nets using the correspondence through source theory. Each such force then provides another window on the underlying one that forms the net itself.

In the most immediate model of color SU_3 symmetry within net theory, each event has not only the two "external" inputs δ already described, which link up into long fibers of the macroscopic vacuum, but also three additional microscopic "internal" inputs which do not. The color group then mixes the internal inputs. This vacuum is not a checkerboard but, a discrete quantum version of Kaluza-Klein theory. More generally, color ought to label a natural trio of distinguishable defects in the vacuum net which are isomorphic but not mixed by SL_2.

In the vacuum net every event has two local SL_2 groups. One is the symmetry of the successors already discussed. The second is a T image of the first, mixing two predecessors. Its existence depends on the condensate structure. In the vacuum net, one combination of these two SL_2 generators survives as the exact global Lorentz symmetry; perhaps this leaves other combinations for approximate internal particle symmetries.

In this approach our goal is a particular vacuum net, and despite its global nature we do not ask where it comes from; after all, it is simpler than the equally global structure it replaces, a spacetime manifold and action functional, whose infinite internal complexity is masked by familiarity.

Fundamental Approach. The deeper question is the nature and origin of dynamical law or laws. In a synchronic quantum theory, the dynamical law is provided from outside the theory, and it is not too difficult to accept it as an eternal absolute element of nature; this would be comfortably consistent with some traditions. But the dynamical law of the synchronic theory is merely the ambient or vacuum field of the diachronic one, a surrogate for the exosystem, and unmistakably a contingent element of nature, not a necessary one. The term ether is perhaps less misleading than vacuum: What seemed to be the eternal global dynamical law is merely the ether, here the vacuum net. Even if we discover its structure by the phenomenological approach described above, we must still discover its local principle. It conflicts with strong locality to take as fundamental any global vacuum net, as much as any field-theoretic action principle, not matter how accurate it may be for some purposes.

Gauge theories reduce this question enormously by assuming that the ether (in the guise of the vacuum field, the Feynman amplitude, and the action principle) is gauge invariant and of low differential order, both principles allied to locality. For gravity, gauge invariance becomes coordinate invariance, with which Hilbert fixes the action (and now the gravitational ether) up to two physical constants G and Λ. Since net theory purports to be still more unified than gauge theories, it should be required to relate these physical constants to c, h, and ƞ. Net theory, however, has natural correspondents for locality and low order, but not for gauge invariance as yet. Thus, the same question is fundamental to both the phenomenological and the fundamental approach: What is the gauge theory of nets? I expect this theory to be an extension of the present theories of relativity along the following lines.

The idea that the dynamical law is itself contingent and evolves [proposed, for example, by C. S. Peirce in his First Flash theory of around 1900; see Peirce (1931–1935)] has long been implicit in standard physics in the concept of external field. For example, Newton's law of motion for the earth is contingent on the absence of strong gravitational waves; if we include the gravitational field in the system, the concept of dynamical law changes, and Newton's law is no longer even a candidate. Similarly, Dirac's equation governs a hydrogen atom only in the absence of external muon beams. With one partition a field is external and subject to no dynamical equations; with another it is internal and evolves dynamically.

In general, quantum theory asserts that the experimenter and the experimentee are parts of an inseparable whole system, insists that nevertheless we must partition this system, and does not tell us how. Call the two parts of this partition the endosystem and exosystem; the endosystem is the entity under study, and the exosystem includes the experimenter, the experimental apparatus, and the relevant external environment. The exosystem is defined for the purposes of synchronic quantum theory by a maximal commutative subalgebra of the algebra of endosystem variables, containing all the variables determined by the exosystem. In classical physics this is a commutative algebra given once for all with the phase space, but in quantum physics it varies from one experimenter to another.

The concept of dynamical law is defined relative to the quantum partition. Call this kind of relativism, third relativity. The first relativity is that of classical theories, including special and general relativity, which fix both the endosystem and the exosystem and merely permute their variables among themselves; while the second relativity is that of quantum theories, which fix the endosystem but transform the exosystem, bringing in different variables. Third relativity transforms both endosystem and exosystem, and thus extends second relativity as second does first. A special case of third relativity is already formulated by Von Neumann in his theory of measurement, when he postulates that the determinations by an exosystem I of an endosystem III do not depend on whether the apparatus II is lumped with I or III. We generalize this to the third relativity principle: The quantum theories for different partitions of the system are consistent.

QET regarded as a universal quantum language does not fix the endosystem, and so seems a reasonable mode of expression to explore the third relativity principle. If the concept of third relativity transformation can be developed to include and unify those of the coordinate group, gauge groups, and renormalization groups, it would not be surprising if third relativity restricted the form of dynamical laws more strongly than first and second.

5. CONCLUSION

Net theory evidently fuses quantum and relativity principles in a fundamental and locally finite way. In a unified theory of the standard kind, the vertical structure of physics is kept more or less intact, and one unifies by a horizontal merger on the top levels. In net theory particles are defects in the vacuum net and their unification is a by-product of an unanticipated vertical merger accomplished by the operator ι, which unites the six levels of quantum field theory (consisting, say, of classical predicate algebra, set theory, topology, and differential geometry, and quantum kinematics and dynamics) into the one of quantum net dynamics. Quantum logic and classical set theory are both reformulated to permit this fusion. My quantum logic fulfills principles of relativistic locality, intensionality, and extensionality that Von Neumann's lacks. My set theory is based on disjoint union and the ι operator instead of union and the $\in$ relation.

To illustrate this fusion of relativity and quantum theory, I develop a trial ψ vector for the vacuum as a quantum condensation of pairs of nodes that supports a Dirac equation for its holes. This vacuum net has exact Lorentz SL_2 invariance, but also an inherent P and T asymmetry. I identify this tentatively with the weak P and T violation. Dimensional grounds (though these are not yet reliable in a theory with so many large dimensionless numbers available) then suggest that the critical temperature T_C should be closer to $M_W \sim 10^2$ GeV, the mass of the W particle, than to the Planck mass $M_P \sim 10^{19}$ GeV, and that balls of a high-temperature phase, Vacuum II, are produced in experiments today as well as in the creation of the universe.

The internal consistency and unusual unity of this fusion of quantum theory and relativity give me confidence that quantum nets provide the correct theory of quantum spin: A spin transformation permutes the two successors to an event among themselves. The "quantum two-valuedness" of Pauli counts the two successors to each event.

APPENDIX

Some notation, especially for indices:

C	complex conjugation
D	dual
Σ	a $D(1/2, 0)$ or Weyl spinor index with values $\uparrow$ and $\downarrow$ (spin up and down)
$*$	labels independent variables cogredient to the complex conjugate
Σ^*	$= \uparrow^*, \downarrow^*$ is a conjugate Weyl spinor index independent of Σ
s	space-time index
S_H	Hermitian-symmetrizes in the index pair $\Sigma^*\Sigma$
σ	$= S_H^{\Sigma^*\Sigma} = \uparrow^*\uparrow,\ \uparrow^*\downarrow,\ \downarrow^*\uparrow,\ \downarrow^*\downarrow$, a real-$D(1/2, 1/2)$ or *sesquispinor* index
(σ)	$= \sigma_1 \ldots \sigma_Z$, a collective index made of $Z\sigma$'s
S_-	average over the symmetric group S_Z on (σ) with weight $+1$ (-1) for even (odd) permutations

$[s]$	$S_{-}\ldots(s)$, a collective antisymmetric index made of $Z\,s$'s
S_{+}	average over the symmetric group S_Z on (σ)
$\{\sigma\}$	$=S_{+}\ldots(\sigma)$, a collective symmetric index made of $Z\,\sigma$'s
$\mathbb{R}^{\sigma}$	real linear space of vectors r^{σ}
$\mathbb{R}^{\{\sigma\}}$	symmetric tensor algebra over $\mathbb{R}^{\sigma}$
α	Dirac or $D(1/2,0)\oplus D(0,1/2)^{D}$ index
$\mathbb{C}^{\alpha}$	complex linear space of Dirac spinors c^{α}
$A(V)$	algebra of linear transformations of the linear space V
$\mathsf{V}(\cdot)$	Grassmann algebra generated by the variables or the linear space $\cdot$

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